

A PRIORI ESTIMATES FOR FLUID INTERFACE PROBLEMS

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ABSTRACT. We consider the regularity of an interface between two incompressible and inviscid fluids flows in the presence of surface tension. We obtain local in time estimates on the interface in $H^{\frac{3}{2}k+1}$ and the velocity fields in $H^{\frac{3}{2}k}$. These estimates are obtained using geometric considerations which show that the Kelvin-Helmholtz instabilities are a consequence of a curvature calculation.

1. INTRODUCTION

In this manuscript we consider the interface problem between two incompressible and inviscid fluids that occupy domains Ω_t^+ and Ω_t^- in \mathbb{R}^n , $n \geq 2$, at time t . We assume that $\mathbb{R}^n = \Omega_t^+ \cup \Omega_t^- \cup S_t$ where $S_t = \partial\Omega_t^\pm$, and let $p_\pm : \Omega_t^\pm \rightarrow \mathbb{R}$, $v_\pm : \Omega_t^\pm \rightarrow \mathbb{R}^n$, and the constant $\rho_\pm > 0$ denote the pressure, the velocity vector field, and the density respectively. On the interface S_t , we let $N_\pm(t, x)$, $x \in S_t$ denote the unit outward normal of Ω_t^\pm (thus $N_+ + N_- = 0$), $H(t, x) \in (T_x S_t)^\perp$ denote the mean curvature vector, and $\kappa_\pm = H \cdot N_\pm$. We also assume that there is surface tension on the interface given by the mean curvature. Thus the free boundary problem for the Euler equation that we consider here is given by

$$(E) \quad \begin{cases} \rho(v_t + \nabla_v v) = -\nabla p, & x \in \mathbb{R}^n \setminus S_t \\ \nabla \cdot v = 0, & x \in \mathbb{R}^n \setminus S_t, \end{cases}$$

The boundary conditions for the interface evolution and the pressure are

$$(BC) \quad \begin{cases} \partial_t + v_\pm \cdot \nabla \text{ is tangent to } \bigcup_t S_t \subset \mathbb{R}^{n+1}, \\ p_+(t, x) - p_-(t, x) = \kappa_+(t, x), & x \in S_t. \end{cases}$$

where we introduced the notation $v = v_+ \mathbf{1}_{\Omega_+} + v_- \mathbf{1}_{\Omega_-} : \mathbb{R}^n \setminus S_t \rightarrow \mathbb{R}^n$, etc.

The boundary conditions (BC) are a consequence of assuming that 1) the interface velocity is given by the normal component of the velocity $v_\pm^\perp = v_\pm \cdot N_\pm$, and that 2) the surface tension on the interface S_t is given by the mean curvature of the surface. A weak formulation for the Euler flow with this form of surface tension is given by

$$\begin{cases} (\rho v)_t + \nabla_v \rho v = -\nabla p + H(t, x) \delta(S_t), & x \in \mathbb{R}^n \\ \nabla \cdot v = 0, & x \in \mathbb{R}^n, \end{cases}$$

where δ is the Dirac mass distribution. This weak formulation implies the boundary condition for the pressure stated in (BC).

Here we consider the problem where Ω^+ is compact and derive a priori estimates, local in time, to the problem (E, BC) that prove bounds on $v(t, \cdot) \in H^{\frac{3}{2}k}(\mathbb{R}^n \setminus S_t)$ and $S_t \in H^{\frac{3}{2}k+1}$ for $\frac{3}{2}k > \frac{n}{2} + 1$. The assumption that Ω^+ is compact is not necessary. In fact the same estimates hold if we assume that S_t is either periodic or asymptotically flat. A more interesting observation is that our proof works verbatim for the case where there are several fluids occupying regions Ω_t^i with interfaces S_t^i . Thus we can treat more general setting than the existing literature.

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The interface problem between two fluids has been studied extensively in the math and physics literature. In the absence of surface tension it is well known that the interface problem between two inviscid and incompressible fluids is ill-posed due to the Kelvin-Helmholtz instability, and it is argued on physical basis that the surface tension is a regularizing force that should make the problem well posed. In [BHL93], Beal, Hou, and Lowengrub demonstrated that the surface tension makes the linearized problem well-posed. For the full nonlinear problem rigorous results have been obtained for irrotational velocities. In this case the problem can be completely reduced to the interface evolution with nonlocal operators. For this problem, Iguchi, Tanaka, and Tani [ITT97] proved the local well-posedness in 2 dimensions with initial interface almost flat and initial velocity almost zero. For the general irrotational problem, Ambrose [AM03] and, more recently, Ambrose and Masmoudi [AM06] proved the local well-posedness in 2 and 3 dimensions, respectively.

We should note that for irrotational flows without surface tension the interface problem is given by the Birkhoff-Rott differential-integral equation. Several results were obtained in this case such as those obtained by Sulem, Sulem, Bardos, and Frisch [SSBF81], and Wu [Wu06]. We also note that without surface tension one can consider the Euler equation with discontinuous velocity fields. Although the interface problem is ill-posed, due to the Kelvin-Helmholtz instability, weak solutions to the Euler equation which may include such discontinuities have been considered by DiPerna and Majda [DM87], Delort [De91], and others. There is also a rich literature of numerical studies of the interface problem, see for example [HLS97] and references therein.

A related problem to the interface problem is the water wave problem where there is only one fluid. In this case the Rayleigh-Taylor instability, instead of the Kelvin-Helmholtz instability, may occur. Such problems have been extensively studied and there is a vast literature on this subject, see for example, [Wu97], [Wu99], [Li05].

Our approach in obtaining energy estimate for the interface problem is similar to the water waves problem treated in [SZ06] in that it is geometric in nature. It is based on the well known fact that these free boundary problems have a variational formulation on a subspace of volume preserving homeomorphisms. We use this variational approach to determine the terms that should be included in the energy. Of course these terms are identified as being the highest order terms of the linearized problem to (E, BC). It is worth noting that from our analysis of the operators involved in the linearized problem, the Kelvin-Helmholtz instability appears naturally as a consequence of the negative semi-definiteness of the leading part of the unbounded curvature operator of the infinite dimensional manifold of admissible Lagrangian coordinate maps. The surface tension, created by the potential energy of the surface area, generates a higher order positive operator that makes the linear problem well-posed and help to establish the energy estimates. The well-posedness of the full problem will be addressed in a forthcoming article.

Our paper is organized as follows. In section 2 we explain how to determine the pressure from the velocity. In section 3 we give a variational formulation of the problem as a constrained variational problem for volume preserving maps. We use this formulation to motivate our definition of energy. In section 4 we prove that our energy controls the Sobolev norm of the velocity and the mean curvature and derive bounds on the energy. Some of the details in the geometric calculations are omitted since they are given in details in [SZ06] and are available as notes on the web at <http://www.math.gatech.edu/~zengch/notes/notes1.pdf>.

Notation All notations will be defined as they are introduced. In addition a list of symbols will be given at the end of the paper for a quick reference. The regularity of the domains Ω_t^\pm is characterized by the local regularity of S_t as graphs. In general, an m -dimensional manifold $\mathcal{M} \subset \mathbb{R}^n$ is said to be of class C^k or H^s , $s > \frac{m}{2}$, if, locally in linear frames, \mathcal{M} can be represented by graphs of C^k or H^s mappings, respectively.

As in [SZ06] Δ_\pm^{-1} denote the inverse Laplacian with zero Dirichlet data, \mathcal{H}_\pm denote the harmonic extension of functions defined on S_t into Ω_t^\pm , and \mathcal{N}_\pm denote the Dirichlet to Neuman operators in the domain Ω_t^\pm . Given two fluids in Ω_t^\pm with constant densities ρ_\pm we denote by $\mathcal{N} = \frac{1}{\rho_+}\mathcal{N}_+ + \frac{1}{\rho_-}\mathcal{N}_-$,

the operator \mathcal{N}^{-1} acts on function with mean zero and its range are also functions with mean zero. For any quantity q defined on $\mathbb{R}^n \setminus S_t$ we write $q = q_+ \mathbf{1}_{\Omega_+} + q_- \mathbf{1}_{\Omega_-}$ where $q_{\pm} = q|_{\Omega_{\pm}}$.

2. DETERMINING THE PRESSURE

In this section we explain how to express the pressure in terms of the velocity in this setting which is less clear than the free boundary problem of water wave in vacuum where the boundary conditions of p_{\pm} are obvious.

To determine the boundary value of p_{\pm} we take the dot product of Euler's equation (E) with N_{\pm}

$$-N_{\pm} \cdot \nabla p_{\pm} = \rho_{\pm} \mathbf{D}_{t\pm}(v_{\pm} \cdot N_{\pm}) - \rho_{\pm} v_{\pm} \cdot \mathbf{D}_{t\pm} N_{\pm}$$

and using the fact that $v_+^{\perp} + v_-^{\perp} = 0$, we obtain

$$\frac{1}{\rho_+} \nabla_{N_+} p_+ + \frac{1}{\rho_-} \nabla_{N_-} p_- = v_+ \cdot \mathbf{D}_{t+} N_+ + v_- \cdot \mathbf{D}_{t-} N_- - \nabla_{v_+^{\perp} - v_-^{\perp}} v_+^{\perp}.$$

Substituting the formula for $\mathbf{D}_{t\pm} N_{\pm}$, which has been calculated in [SZ06],

$$(2.1) \quad \mathbf{D}_{t\pm} N_{\pm} = -((Dv_{\pm})^*(N_{\pm}))^{\top}$$

we have

$$\frac{1}{\rho_+} \nabla_{N_+} p_+ + \frac{1}{\rho_-} \nabla_{N_-} p_- = \Pi_+(v_+^{\top}, v_+^{\top}) + \Pi_-(v_-^{\top}, v_-^{\top}) - 2\nabla_{v_+^{\perp} - v_-^{\perp}} v_+^{\perp},$$

where Π_{\pm} is the second fundamental form of S_t associated to N_{\pm} , which satisfy $\Pi_+ + \Pi_- = 0$. Since $p_{\pm} = \mathcal{H}_{\pm}(p_{\pm}|_{S_t}) + \Delta_{\pm}^{-1} \Delta p_{\pm}$ in Ω_{\pm}^{\pm} , we have

$$\begin{aligned} \frac{1}{\rho_+} \mathcal{N}_+ p_+ + \frac{1}{\rho_-} \mathcal{N}_- p_- &= -\frac{1}{\rho_+} \nabla_{N_+} \Delta_+^{-1} \Delta p_+ - \frac{1}{\rho_-} \nabla_{N_-} \Delta_-^{-1} \Delta p_- \\ &\quad + \Pi_+(v_+^{\top}, v_+^{\top}) + \Pi_-(v_-^{\top}, v_-^{\top}) - 2\nabla_{v_+^{\perp} - v_-^{\perp}} v_+^{\perp} \quad \text{on } S_t. \end{aligned}$$

The boundary condition $p_+ - p_- = \kappa_+$ on S_t stated in (BC) implies that on S_t

$$\begin{aligned} p_{\pm} &= \mathcal{N}^{-1} \left(-\frac{1}{\rho_{\mp}} \mathcal{N}_{\mp} \kappa_{\mp} - \frac{1}{\rho_+} \nabla_{N_+} \Delta_+^{-1} \Delta p_+ - \frac{1}{\rho_-} \nabla_{N_-} \Delta_-^{-1} \Delta p_- \right) \\ &\quad + \Pi_+(v_+^{\top}, v_+^{\top}) + \Pi_-(v_-^{\top}, v_-^{\top}) - 2\nabla_{v_+^{\perp} - v_-^{\perp}} v_+^{\perp}. \end{aligned}$$

Finally, since $\nabla \cdot v = 0$ in $\mathbb{R}^n \setminus S_t$, we have from (E)

$$(2.2) \quad -\Delta p = \rho \nabla \cdot (\nabla v v) = \rho \operatorname{tr}(Dv)^2, \quad x \in \mathbb{R}^n \setminus S_t.$$

Therefore,

$$(2.3) \quad \begin{aligned} p_{\pm}|_{S_t} &= \mathcal{N}^{-1} \left(-\frac{1}{\rho_{\mp}} \mathcal{N}_{\mp} \kappa_{\mp} + \nabla_{N_+} \Delta_+^{-1} \operatorname{tr}(Dv)^2 + \nabla_{N_-} \Delta_-^{-1} \operatorname{tr}(Dv)^2 \right) \\ &\quad + \Pi_+(v_+^{\top}, v_+^{\top}) + \Pi_-(v_-^{\top}, v_-^{\top}) - 2\nabla_{v_+^{\perp} - v_-^{\perp}} v_+^{\perp}. \end{aligned}$$

One can verify that the quantity \mathcal{N}^{-1} acts on in the above has zero mean on S_t and thus p is well defined by (2.2) and (2.3).

3. LAGRANGIAN FORMULATION AND THE ENERGY

This section is intended to explain the intuition behind the energy. It illustrates how to isolate the leading order nonlinear terms \mathcal{A} and \mathcal{R}_0 defined in (3.22) and (3.25) respectively.

In his 1966 seminal paper [Ar66], V. Arnold pointed out that the Euler equation for an incompressible inviscid fluid can be viewed as the geodesic equation on the group of volume preserving diffeomorphisms. This point of view has been adopted and developed by several authors such as D. G. Ebin and G. Marsden [EM70], A. Shnirelman [Sh85], and Y. Brenier [Br99], to mention a few, in their work on Euler's equations on fixed domains. It is this point of view that we adopted

to explain the motivation for our definition of energy for the water waves problem [SZ06], and it is this same point of view that forms our starting point to determine the appropriate energy for the interface problem .

3.1. Lagrangian formulation of the problem. Conservation of energy can be obtained from multiplying the Euler's equation (E) by v , integrating on $\mathbb{R}^n \setminus S_t$, and using (BC) to obtain the conserved energy E_0 :

$$(3.1) \quad E_0 = E_0(S_t, v) = \int_{\mathbb{R}^n} \frac{\rho|v|^2}{2} dx + \int_{S_t} dS \triangleq \int_{\Omega_t} \frac{\rho|v|^2}{2} dx + S(S_t),$$

where $S(\cdot)$ denotes the surface area.

Let $u_{\pm}(t, y)$, $y \in \Omega_0^{\pm}$, be the Lagrangian coordinate map solving

$$(3.2) \quad \frac{dx}{dt} = v(t, x), \quad x(0) = y,$$

then we have $v = u_t \circ u^{-1}$, and for any vector field w on $x \in \mathbb{R}^n \setminus S_t$, $\mathbf{D}_t w = (w \circ u)_t \circ u^{-1}$. Therefore in Lagrangian coordinates the Euler's equation takes the form

$$(3.3) \quad \rho u_{tt} = -(\nabla p) \circ u, \quad u(0) = id_{\Omega_0},$$

where the pressure p is given by (2.2) and (2.3).

Since $v(t, \cdot)$ is divergence free in $\mathbb{R}^n \setminus S_t$, then $u_{\pm}(t, \cdot)$ are volume preserving. Moreover, while $u_+(t, \cdot)|_{S_0} = u_-(t, \cdot)|_{S_0}$ may not hold, it is clear that $u_+(t, S_0) = u_-(t, S_0)$. Thus the Lagrangian coordinates maps satisfy:

- 1) $\Phi_{\pm} : \bar{\Omega}^{\pm} \rightarrow \Phi_{\pm}(\bar{\Omega}^{\pm})$ a volume preserving homeomorphism.
- 2) $S \triangleq \partial\Phi_{\pm}(\Omega^{\pm}) = \Phi(\partial\Omega^{\pm})$

Define

$$\Gamma = \{\Phi = \Phi_+ \mathbb{1}_{\Omega^+} + \Phi_- \mathbb{1}_{\Omega^-}; \quad \Phi_{\pm} \text{ satisfy 1 and 2 above}\}.$$

As a manifold, the tangent space of Γ is given by divergence free vector fields with matching normal component in Eulerian coordinates:

$$T_{\Phi}\Gamma = \{\bar{w} : \mathbb{R}^n \setminus S_0 \rightarrow \mathbb{R}^n \mid \nabla \cdot w = 0 \text{ and } w_{\perp}^+ + w_{\perp}^-|_{\Phi(S_0)} = 0, \text{ where } w = (\bar{w} \circ \Phi^{-1})\}.$$

Here as in [SZ06] we are following the convention that for any vector field $X : \Phi(\Omega_0) \rightarrow \mathbb{R}^n$ its description in Lagrangian coordinates is given by $\bar{X} = X \circ \Phi$.

Writing $S(\Phi) = \int_{\Phi(S_0)} dS$ for the surface area of $\Phi(S_0)$, the energy E_0 in Lagrangian coordinates can be written as:

$$(3.4) \quad E_0 = E_0(u, u_t) = \frac{1}{2} \int_{\mathbb{R}^n \setminus S_0} \rho |u_t|^2 dy + S(u), \quad (u, u_t) \in T\Gamma$$

where the volume preserving property of u is used. This conservation of energy suggests: 1) $T\Gamma$ be endowed with the $L^2(\rho dy)$ metric*; and 2) the free boundary problem of the Euler's equation has a Lagrangian action

$$I(u) = \int \int_{\mathbb{R}^n \setminus S_0} \frac{\rho |u_t|^2}{2} dy dt - \int S(u) dt, \quad u(t, \cdot) \in \Gamma.$$

Let $\bar{\mathcal{D}}$ denote the covariant derivative associated with the metric on Γ , then a critical path $u(t, \cdot)$ of I satisfies

$$(3.5) \quad \bar{\mathcal{D}}_t u_t + S'(u) = 0.$$

In order to verify that the Lagrangian coordinate map $u(t, \cdot)$ satisfying (E) and (BC) is indeed a critical path of I , it is convenient to calculate $\bar{\mathcal{D}}$ and S' by viewing Γ as a submanifold of the Hilbert space $L^2(\mathbb{R}^n \setminus S_0, \rho dy, \mathbb{R}^n)$.

*Including the density in the volume element ρdy introduces a factor of $\frac{1}{\rho}$ in front of the physical pressure.

$(T_\Phi \Gamma)^\perp$ and orthogonal decomposition of vector fields. For any vector field X defined on $\Phi(\mathbb{R}^n \setminus S_0)$, Hodge decomposition suggests that we decompose X into $X = w - \nabla \psi$, with $\psi = \psi_+ \mathbf{1}_{\Omega_+} + \psi_- \mathbf{1}_{\Omega_-}$, so that $\bar{w} = w \circ \Phi \in T_\Phi \Gamma$ and $\nabla \psi \circ \Phi \in (T_\Phi \Gamma)^\perp$. For any $\bar{Y} \in T_\Phi \Gamma$, the orthogonality $\int \nabla \psi \cdot Y \rho dy = 0$ implies

$$\int_{\Phi(S_0)} \rho_+ \psi_+ Y_+^\perp + \rho_- \psi_- Y_-^\perp dS = 0.$$

Therefore, ψ must satisfy $\rho_+ \psi_+ = \rho_- \psi_- \triangleq \psi^S$ on $\Phi(S_0)$. This suggests that, for any $\Phi \in \Gamma$,

$$(T_\Phi \Gamma)^\perp = \{-(\nabla \psi) \circ \Phi \mid \rho_+ \psi_+ = \rho_- \psi_- \text{ on } \Phi(S_0)\}.$$

To prove this claim, we only need to find such a ψ given X . From $X = w - \nabla \psi$ and $w_+^\perp + w_-^\perp = 0$, we have

$$(3.6) \quad X_+^\perp + X_-^\perp = -\nabla_{N_+} \psi_+ - \nabla_{N_-} \psi_- = -\mathcal{N} \psi^S - \nabla_{N_+} \Delta_+^{-1} \Delta \psi - \nabla_{N_-} \Delta_-^{-1} \Delta \psi.$$

Since $\nabla \cdot w = 0$, we obtain

$$(3.7) \quad \begin{cases} -\Delta \psi = \nabla \cdot X \\ \psi_\pm|_{\Phi(S_0)} = \frac{1}{\rho_\pm} \psi^S = -\frac{1}{\rho_\pm} \mathcal{N}^{-1} (X_+^\perp + X_-^\perp - \nabla_{N_+} \Delta_+^{-1} \nabla \cdot X - \nabla_{N_-} \Delta_-^{-1} \nabla \cdot X). \end{cases}$$

It is easy to verify that $w = X - \nabla \psi$ satisfies $\bar{w} = w \circ \Phi \in T_\Phi \Gamma$.

Computing $\bar{\mathcal{D}}_t$ and II_Φ . Given a path $u(t, \cdot) \in \Gamma$ and $\bar{v} = u_t$. Let $S_t = u(t, S_0)$. Suppose $\bar{w}(t, \cdot) \in T_{u(t)} \Gamma$, then the covariant derivative $\bar{\mathcal{D}}_t \bar{w}$ and the second fundamental form $II_{u(t)}(\bar{w}, \bar{v})$ satisfy

$$\bar{w}_t = \bar{\mathcal{D}}_t \bar{w} + II_{u(t)}(\bar{w}, \bar{v}), \quad \bar{\mathcal{D}}_t \bar{w} \in T_{u(t)} \Gamma, \quad II_{u(t)}(\bar{w}, \bar{v}) \in (T_{u(t)} \Gamma)^\perp.$$

Let $v = u_t \circ u^{-1} = \bar{v} \circ u^{-1}$ and $w = \bar{w} \circ u^{-1}$ be the Eulerian coordinates description of u_t and w , then for $X = \mathbf{D}_t w$ there exists $p_{v,w} = p_{w,v}^+ \mathbf{1}_{\Omega_+} + p_{w,v}^- \mathbf{1}_{\Omega_-} : \mathbb{R}^n \setminus u(t, S_0) \rightarrow \mathbb{R}$ determined by (3.7) such that

$$(3.8) \quad \rho_+ p_{v,w}^+ = \rho_- p_{v,w}^- \text{ on } S_t, \quad II(\bar{w}, \bar{v}) = -(\nabla p_{w,v}) \circ u \in (T_{u(t)} \Gamma)^\perp.$$

The Eulerian coordinates description of the covariant derivative is given by

$$(3.9) \quad \mathcal{D}_t w = (\bar{\mathcal{D}}_t \bar{w}) \circ u^{-1} = \mathbf{D}_t w + \nabla p_{w,v}.$$

The terms involving $X = \mathbf{D}_t w$ in equation (3.7) are expressed as follows. From $w_+^\perp + w_-^\perp = 0$ on $u(t, S_0)$ and identity (2.1) we have

$$(3.10) \quad \begin{aligned} (\mathbf{D}_{t_+} w_+) \cdot N_+ + (\mathbf{D}_{t_-} w_-) \cdot N_- &= \mathbf{D}_{t_+} w_+^\perp + \mathbf{D}_{t_-} w_-^\perp + \nabla_{w_+^\perp} v_+ \cdot N_+ + \nabla_{w_-^\perp} v_- \cdot N_- \\ &= \nabla_{v_+^\perp - v_-^\perp} w_+^\perp + \nabla_{w_+^\perp - w_-^\perp} v_+^\perp - \Pi_+(v_+^\perp, w_+^\perp) - \Pi_-(v_-^\perp, w_-^\perp). \end{aligned}$$

And since $\nabla \cdot \mathbf{D}_t w = \text{tr}(Dv Dw)$ then $p_{w,v}$ is given by

$$(3.11) \quad \begin{cases} -\Delta p_{w,v} = \text{tr}(Dv Dw) \\ p_{w,v}^\pm|_{S_t} = \frac{1}{\rho_\pm} p_{w,v}^S = -\frac{1}{\rho_\pm} \mathcal{N}^{-1} \{ \nabla_{v_+^\perp - v_-^\perp} w_+^\perp + \nabla_{w_+^\perp - w_-^\perp} v_+^\perp - \Pi_+(v_+^\perp, w_+^\perp) \\ \quad - \Pi_-(v_-^\perp, w_-^\perp) - \nabla_{N_+} \Delta_+^{-1} \text{tr}(Dv Dw) - \nabla_{N_-} \Delta_-^{-1} \text{tr}(Dv Dw) \} \end{cases}$$

A more useful way to express the boundary value $p_{w,v}^S$ is as follows. From the divergence decomposition formula

$$(3.12) \quad 0 = \nabla \cdot v_\pm = \mathcal{D} \cdot v_\pm^\top + \kappa_\pm v_\pm^\perp + N_\pm \cdot \nabla_{N_\pm} v_\pm \quad \text{on } S_t,$$

where \mathcal{D} is the covariant derivative on S_t , implies

$$(3.13) \quad \nabla_{w_\pm} v_\pm \cdot N_\pm = \nabla_{w_\pm^\perp} v_\pm \cdot N_\pm - \kappa_\pm w_\pm^\perp v_\pm^\perp - \mathcal{D} \cdot (w_\pm^\perp v_\pm^\top) + \nabla_{v_\pm^\perp} w_\pm^\perp.$$

Thus, we have

$$(3.14) \quad \begin{aligned} p_{w,v}^S = & -\mathcal{N}^{-1}\{\nabla_{w_+} v_+ \cdot N_+ + \nabla_{w_-} v_- \cdot N_- + \mathcal{D} \cdot (w_+^\perp (v_+^\top - v_-^\top)) \\ & - \nabla_{N_+} \Delta_+^{-1} \text{tr}(Dv Dw) - \nabla_{N_-} \Delta_-^{-1} \text{tr}(Dv Dw)\}. \end{aligned}$$

Moreover, for any smooth function f defined on S_t , we have from the Divergence theorem,

$$\int_{S_t} -f \nabla_{N_\pm} \Delta_\pm^{-1} \text{tr}(Dv_\pm Dw_\pm) dS = - \int_{\Omega_t^\pm} \nabla f \mathcal{H}_\pm \cdot \nabla \Delta_\pm^{-1} \text{tr}(Dv_\pm Dw_\pm) + f \mathcal{H}_\pm \text{tr}(Dv_\pm Dw_\pm) dx$$

Again, by the Divergence Theorem the first term integrates to zero and the second term can be written as

$$(3.15) \quad \begin{aligned} \int_{S_t} -f \nabla_{N_\pm} \Delta_\pm^{-1} \text{tr}(Dv_\pm Dw_\pm) dS = & \int_{S_t} -f \nabla_{w_\pm} v_\pm \cdot N_\pm + w_\pm^\perp \nabla f \mathcal{H}_\pm \cdot v_\pm dS \\ & - \int_{\Omega_t^\pm} D^2 f \mathcal{H}_\pm(v_\pm, w_\pm) dx. \end{aligned}$$

Thus, using the decomposition $\nabla f \mathcal{H}_\pm = \nabla^\top f + (\mathcal{N}_\pm f) N_\pm$ and letting $f = -\mathcal{N}^{-1}g$, we obtain

$$(3.16) \quad \int_{S_t} g p_{w,v}^S dS = \int_{S_t} -w_+^\perp v_+^\perp (\mathcal{N}_+ + \mathcal{N}_-) \mathcal{N}^{-1} g dS + \int_{\mathbb{R}^n \setminus S_t} D^2(\mathcal{H}_\pm(\mathcal{N}^{-1}g))(v, w) dx.$$

Computing $S'(\Phi)$. By the variation of surface area formula, for any $\bar{w} \in T_\Phi \Gamma$ we have

$$\langle S'(\Phi), \bar{w} \rangle_{L^2(\mathbb{R}^n \setminus S_0, \rho dy)} = \int_{\Phi(S_0)} \kappa_+ w_+^\perp dS = \int_{\Phi(S_0)} \kappa_- w_-^\perp dS.$$

We need to find the unique representation $S'(\Phi)$ in $T_\Phi \Gamma$ of the above functional.

Lemma 3.1. *For any smooth function $f_0 : \Phi(S_0) \rightarrow \mathbb{R}$, let $f_\pm = \pm \frac{1}{\rho_+ \rho_-} \mathcal{H}_\pm \mathcal{N}^{-1} \mathcal{N}_\mp f_0$ and $f = f_+ \mathbf{1}_{\Omega^+} + f_- \mathbf{1}_{\Omega^-}$, then we have $\nabla f \in T_\Phi \Gamma$ and for any $\bar{w} \in T_\Phi \Gamma$,*

$$\int_{\Phi(S_0)} f_0 w_+^\perp dS = \int_{\mathbb{R}^n \setminus \Phi(S_0)} w \cdot \nabla f \rho dx.$$

The verification of the lemma is straightforward. Therefore, we have

$$(3.17) \quad S'(\Phi) = \nabla p_\kappa \quad \text{where } p_\kappa^\pm = \frac{1}{\rho_+ \rho_-} \mathcal{H}_\pm \mathcal{N}^{-1} \mathcal{N}_\mp \kappa_\pm.$$

Splitting of the pressure. From (2.3), (3.7) and (3.17), it is clear that $\rho(p_{v,v} + p_\kappa) = p$. Therefore, we obtain the well known equivalence between equation (3.5) for critical paths of I and the Euler's equation (E) with the free boundary condition (BC). The Euler' equation can also be written as

$$(3.18) \quad \mathbf{D}_t v + \nabla p_{v,v} + \nabla p_\kappa = 0.$$

Notice that the pressure splits into two terms, the first $p_{v,v}$ is the Lagrange multiplier, and the second p_κ is due to surface tension. These two terms will be treated differently in the energy estimates.

3.2. Linearization. In order to analyze the free boundary problems of the Euler's equation, it is natural to start with the linearization. The Lagrangian formulation provides a convenient frame work for this purpose. From (3.5), the linearized equation is

$$(3.19) \quad \bar{\mathcal{D}}_t^2 \bar{w} + \bar{\mathcal{R}}(u_t, \bar{w}) u_t + \bar{\mathcal{D}}^2 S(u)(\bar{w}) = 0, \quad \bar{w}(t, \cdot) \in T_{u(t, \cdot)} \Gamma,$$

where $\bar{\mathcal{R}}$ is the curvature tensor of the infinite dimensional manifold Γ . Below we calculate $\bar{\mathcal{R}}$ and $\bar{\mathcal{D}}^2 S(u)$, which is a linear operator on $T_u \Gamma$. Since these operators are self-adjoint, we will compute their quadratic forms.

Computing $\bar{\mathcal{D}}^2 S(u)$. The formula for $\bar{\mathcal{D}}^2 S(u)$ was given in [SZ06]

$$\begin{aligned} \bar{\mathcal{D}}^2 S(u)(\bar{w}, \bar{w}) &= \int_{S_t} \kappa_{\pm} w_{\pm}^{\perp} (\kappa_{\pm} w_{\pm}^{\perp} + \mathcal{D} \cdot w_{\pm}^{\top}) - \kappa_{\pm} \nabla_{N_{\pm}} p_{w,w}^{\pm} - \kappa_{\pm} \nabla_{w_{\pm}^{\top}} w_{\pm} \cdot N_{\pm} \\ &\quad + w_{\pm}^{\perp} \left(-\Delta_{S_t} w_{\pm}^{\perp} - w_{\pm}^{\perp} |\Pi|^2 + \nabla_{w_{\pm}^{\top}} \kappa_{\pm} \right) dS, \end{aligned}$$

for any $\bar{w} \in T_u \Gamma$, where \mathcal{D} is the Riemannian connection and Δ_{S_t} is the Beltrami-Lapalacian operator on S_t . Of course $\bar{\mathcal{D}}^2 S(u)(\bar{w}, \bar{w})$ is independent of the choice of the $+$ or $-$ sign.

Needless to say that this is a very complicated expression for $\bar{\mathcal{D}}^2 S(u)(\bar{w}, \bar{w})$. We will single out its leading order part. Since the value of $\bar{\mathcal{D}}^2 S(u)$ does not depend on the choice of $+$ or $-$ sign, we compute with the $+$ sign and assume that S_t is a sufficiently smooth hypersurface. From the Divergence Theorem,

$$\begin{aligned} &|\bar{\mathcal{D}}^2 S(u)(\bar{w}, \bar{w}) - \int_{S_t} |\nabla^{\top} w_{+}^{\perp}|^2 dS| \\ &\leq \left| \int_{S_t} \kappa_{+} (\nabla_{N_{+}} p_{w,w}^{+} + \nabla_{w_{+}^{\top}} w_{+} \cdot N_{+} + \nabla_{w_{+}^{\top}} w_{+}^{\perp}) dS \right| + C|w|_{L^2(S_t)}^2. \end{aligned}$$

To estimate the integral on the right side, we use the splitting of $\nabla_{N_{+}} p_{w,w}^{+}$ on S_t

$$\nabla_{N_{+}} p_{w,w}^{+} = \mathcal{N}_{+}(p_{w,w}^{+}|_{S_t}) + \nabla_{N_{+}} \Delta_{+}^{-1} \Delta p_{w,w}^{+} = \frac{1}{\rho_{+}} \mathcal{N}_{+} p_{w,w}^S - \nabla_{N_{+}} \Delta_{+}^{-1} \text{tr}(Dw)^2.$$

and identities (3.15) and (3.13) to obtain

$$|\bar{\mathcal{D}}^2 S(u)(\bar{w}, \bar{w}) - \int_{S_t} |\nabla^{\top} w_{+}^{\perp}|^2 dS| \leq \left| \int_{S_t} p_{w,w}^S \frac{1}{\rho_{+}} \mathcal{N}_{+} \kappa_{+} dS \right| + C(|w|_{L^2(S_t)}^2 + |w|_{L^2(\mathbb{R}^n \setminus S_t)}^2).$$

Finally, from (3.16), we have

$$(3.20) \quad |\bar{\mathcal{D}}^2 S(u)(\bar{w}, \bar{w}) - \int_{S_t} |\nabla^{\top} w_{+}^{\perp}|^2 dS| \leq C(|w|_{L^2(S_t)}^2 + |w|_{L^2(\mathbb{R}^n \setminus S_t)}^2).$$

Using Lemma 3.1, we can define a self-adjoint positive semi-definite operator $\bar{\mathcal{A}}(u)$ on $T_u \Gamma$ as the leading order part of $\bar{\mathcal{D}}^2 S(u)$. In the Eulerian coordinates, $\bar{\mathcal{A}}(u)$ takes the form

$$(3.21) \quad \bar{\mathcal{A}}(u)(w) = \nabla f_{+} \mathbf{1}_{\Omega^{+}} + \nabla f_{-} \mathbf{1}_{\Omega^{-}}, \quad \text{where } f_{\pm} = \frac{1}{\rho_{+} \rho_{-}} \mathcal{H}_{\pm} \mathcal{N}^{-1} \mathcal{N}_{\mp} (-\Delta_{S_t}) w_{\pm}^{\perp}.$$

Clearly $\bar{\mathcal{A}}(u)$ is self-adjoint and satisfies

$$(3.22) \quad \bar{\mathcal{A}}(\bar{w}, \bar{w}) = \int_{S_t} |\nabla^{\top} w_{+}^{\perp}|^2 dS.$$

which is like a third order differential operator on $\mathbb{R}^n \setminus S_t$. From (3.23), we can write

$$(3.23) \quad \bar{\mathcal{D}}^2 S(u) = \bar{\mathcal{A}}(u) + \text{at most 1st order diff. operators}$$

Computing $\bar{\mathcal{R}}$. For any $\bar{v}, \bar{w} \in T_u \Gamma$, let $v = \bar{v} \circ u^{-1}$ and $w = \bar{w} \circ u^{-1}$ we have

$$\bar{\mathcal{R}}(u)(\bar{v}, \bar{w}) \bar{v} \cdot \bar{w} = II_u(\bar{v}, \bar{v}) \cdot II_u(\bar{w}, \bar{w}) - II_u(\bar{v}, \bar{w})^2 = \int_{\mathbb{R}^n \setminus S_t} \rho \nabla p_{v,v} \nabla p_{w,w} - \rho |\nabla p_{v,w}|^2 dx.$$

Assuming that the hypersurface S_t and \bar{v} are sufficiently smooth we single out the leading order term of $\bar{\mathcal{R}}(u)(\bar{v}, \cdot) \bar{v}$.

We first estimate $II_u(\bar{v}, \bar{v}) \cdot II_u(\bar{w}, \bar{w})$. From the Divergence Theorem and (3.11),

$$\int_{\mathbb{R}^n \setminus S_t} \rho \nabla p_{v,v} \nabla p_{w,w} dx = \int_{S_t} p_{v,v}^S (\nabla_{N_{+}} p_{w,w}^{+} + \nabla_{N_{-}} p_{w,w}^{-}) dS + \int_{\mathbb{R}^n \setminus S_t} \rho p_{v,v} \text{tr}(Dw)^2 dx$$

Using (3.6) and (3.10) for the above boundary integral and applying the Divergence Theorem twice for the interior integral, we obtain

$$\begin{aligned} \int_{\mathbb{R}^n \setminus S_t} \rho \nabla p_{v,v} \nabla p_{w,w} dx &= \int_{\mathbb{R}^n \setminus S_t} \rho D^2 p_{v,v}(w, w) dx + \int_{S_t} p_{v,v}^S \{ -2 \nabla_{w_+^\top - w_-^\top} w_+^\perp + \Pi_+(w_+^\top, w_+^\top) \\ &\quad + \Pi_-(w_-^\top, w_-^\top) + \nabla_{w_+} w_+ \cdot N_+ + \nabla_{w_-} w_- \cdot N_- \} - \rho_+ w_+^\perp \nabla_{w_+} p_{v,v}^+ - \rho_- w_-^\perp \nabla_{w_-} p_{v,v}^- dS \end{aligned}$$

Using (3.13) to compute $\nabla_{w_\pm} w_\pm \cdot N_\pm$, we obtain

$$| \int_{\mathbb{R}^n \setminus S_t} \rho \nabla p_{v,v} \nabla p_{w,w} dx | \leq C(|w|_{L^2(S_t)}^2 + |w|_{L^2(\mathbb{R}^n \setminus S_t)}^2).$$

To compute $II_u(\bar{v}, \bar{w})^2$, we use the decomposition $p_{v,w}^\pm = \frac{1}{\rho_\pm} \mathcal{H}_\pm p_{v,w}^S - \Delta_\pm^{-1} \text{tr}(Dv Dw)$ and the the Divergence Theorem to obtain

$$\int_{\mathbb{R}^n \setminus S_t} \rho |\nabla p_{v,w}|^2 dx = \int_{S_t} p_{v,w}^S \mathcal{N} p_{v,w}^S dS + \int_{\mathbb{R}^n \setminus S_t} \rho |\nabla \Delta_\pm^{-1} \text{tr}(Dv Dw)|^2 dx.$$

Therefore,

$$| \int_{\mathbb{R}^n \setminus S_t} \rho |\nabla p_{v,w}|^2 dx - \int_{S_t} p_{v,w}^S \mathcal{N} p_{v,w}^S dS | \leq C|w|_{L^2(\mathbb{R}^n \setminus S_t)}^2.$$

Moreover, from expression (3.14) of $p_{w,v}^S$, we claim the terms other than $\mathcal{N}^{-1} \mathcal{D} \cdot (w_\pm^\perp v_\pm^\top)$ are of lower order. In fact, since

$$|\nabla_{w_\pm} v_\pm - \nabla \Delta_\pm^{-1} \text{tr}(Dv Dw)|_{L^2(\Omega_t^\pm)} \leq C|w|_{L^2(\Omega^\pm)}$$

and it is divergence free, its normal component on S_t is in $H^{-\frac{1}{2}}(S_t)$ and we have

$$|p_{v,w}^S + \mathcal{N}^{-1} \mathcal{D} \cdot (w_+^\perp (v_+^\top - v_-^\top))|_{H^{\frac{1}{2}}(S_t)} \leq C|w|_{L^2(\mathbb{R}^n \setminus S_t)}.$$

Therefore, summarizing these estimates, we obtain

$$(3.24) \quad |\bar{\mathcal{R}}(u)(\bar{v}, \bar{w}) \bar{v} \cdot \bar{w} + \int_{S_t} |\mathcal{N}^{-\frac{1}{2}} \mathcal{D} \cdot (w_+^\perp (v_+^\top - v_-^\top))|^2 dS| \leq C(|w|_{L^2(S_t)}^2 + |w|_{L^2(\mathbb{R}^n \setminus S_t)}^2).$$

Using Lemma 3.1, we define a self-adjoint positive semi-definite operator $\bar{\mathcal{R}}_0(u)(\bar{v})$ on $T_u \Gamma$ which is the leading order part of $\bar{\mathcal{R}}(u)(\bar{v}, \cdot) \bar{v}$. In Eulerian coordinates, $\bar{\mathcal{R}}_0(\bar{v})$ takes the form

$$(3.25) \quad \begin{cases} \mathcal{R}_0(v)(w) = \nabla f_+ \mathbf{1}_{\Omega^+} + \nabla f_- \mathbf{1}_{\Omega^-}, \\ f_\pm = \frac{1}{\rho_+ \rho_-} \mathcal{H}_\pm \mathcal{N}^{-1} \mathcal{N}_\mp \nabla_{v_+^\top - v_-^\top} \mathcal{N}^{-1} \mathcal{D} \cdot (w_\pm^\perp (v_+^\top - v_-^\top)). \end{cases}$$

Assuming smooth S_t and \bar{v} , from (3.24), we can write

$$\bar{\mathcal{R}}(\bar{v}, \bar{w}) \bar{w} = \bar{\mathcal{R}}_0(\bar{v}) \bar{w} + \text{at most 1st order diff. operators.}$$

Clearly $\bar{\mathcal{R}}_0(\bar{v})$ is a second order negative semi-definite differential operator. Therefore, the linearized Euler's equation (3.19) would be ill-posed if there had been no surface tension, for $\bar{\mathcal{R}}_0(\bar{v})$ would become the leading order term. *This is the Kelvin-Helmholtz instability of vortex-sheets.*

Note that since \mathcal{A} is positive definite and is higher order than the self-adjoint $\mathcal{R}_0(v)$, it is not difficult to see that the linearized problem (3.19) is well-posed. For a priori estimates of the Euler's equation (E) with the boundary condition (BC), the positive semi-definiteness of the leading order part $\mathcal{A}(u)$ of $\mathcal{D}^2 S(u)$ suggests to consider the inner product of (3.5) with $(\mathcal{D}^2 S)^k u_t$ to obtain a priori estimates.

4. MAIN RESULTS

In this section, we will derive local energy estimate. We show that solutions of (E) with boundary condition (BC) are locally bounded in

$$(4.1) \quad v(t, \cdot) \in H^{\frac{3}{2}k}(\mathbb{R}^n \setminus S_t) \quad \text{and} \quad S_t \in H^{\frac{3}{2}k+1},$$

where k is an integer satisfying $\frac{3}{2}k > \frac{n}{2} + 1$ (equivalently $\frac{3}{2}k \geq \frac{n}{2} + \frac{3}{2}$).

Definition of the energies and statements of the theorems. The conserved energy of the Euler's equation is given by $E_0 = \int_{\mathbb{R}^n \setminus S_t} \frac{1}{2} |v|^2 dx + S(S_t)$. Higher order energies are based on the linearized Euler flow and thus involve the differential operators \mathcal{A} defined in (3.21) and \mathcal{D}_t defined in (3.9).

Let $\omega_v : \mathbb{R}^n \rightarrow \mathbb{R}^n$, often written as ω for short, represent the curl a vector field v defined on $\mathbb{R}^n \setminus S_t$, i.e.

$$\omega(X) \cdot Y = \nabla_X v \cdot Y - \nabla_Y v \cdot X$$

for any vector $X, Y \in \mathbb{R}^n$. Viewing ω as a matrix, its entries are $\omega_i^j = \omega(\frac{\partial}{\partial x^i}) \cdot \frac{\partial}{\partial x^j} = \partial_i v^j - \partial_j v^i$.

Definition 4.1. Given domains Ω^\pm with Ω_+ compact and the interface S in $H^{\frac{3}{2}k+1}$ and any vector field $v \in H^{\frac{3}{2}k}(\mathbb{R}^n \setminus S)$ with $v_+^\perp + v_-^\perp|_S = 0$ and $\nabla \cdot v = 0$, define the energy $E(S, v)$, often written as E for short,

$$E = \int_{\mathbb{R}^n \setminus S} \frac{1}{2} |\mathcal{A}^{\frac{k}{2}} v|^2 + \frac{1}{2} |\mathcal{A}^{\frac{k}{2}-\frac{1}{2}} \nabla p_\kappa|^2 dx + |\omega|_{H^{\frac{3}{2}k-1}(\mathbb{R}^n \setminus S)}^2,$$

where p_κ is the pressure due to the surface tension defined in (3.17).

Since the free boundary is evolving, we consider the following type of $H^{\frac{3}{2}k-\frac{1}{2}}$ neighborhoods of hypersurfaces to maintain uniform constants in the energy inequalities.

Definition 4.2. Let $\Lambda = \Lambda(S, \frac{3}{2}k - \frac{1}{2}, \delta)$ be the collection of all hypersurfaces \tilde{S} such that there exists a diffeomorphism $F : S \rightarrow \tilde{S} \subset \mathbb{R}^n$, with $|F - id_S|_{H^{\frac{3}{2}k-\frac{1}{2}}(S)} < \delta$.

Fix $0 < \delta \ll 1$ and let $\Lambda_0 \triangleq \Lambda(S_0, \frac{3}{2}k - \frac{1}{2}, \delta)$. From (3.17), (3.21), and (3.22),

$$(4.2) \quad |p_\kappa|_{H^{s+\frac{1}{2}}(\mathbb{R}^n \setminus S)} \leq C |\kappa|_{H^s(S)}, \quad s \in [\frac{1}{2}, \frac{3}{2}k - \frac{1}{2}]$$

$$(4.3) \quad |\mathcal{A}|_{L(H^s(\mathbb{R}^n \setminus S), H^{s-3}(\Omega))} \leq C, \quad s \in [4 - \frac{3}{2}k, \frac{3}{2}k - 1]$$

where C is uniform in $S \in \Lambda_0$. The next proposition gives bounds on the velocity and mean curvature in terms of the energy E .

Proposition 4.1. For $S \in \Lambda_0$ with $S \in H^{\frac{3}{2}k+1}$, we have

$$|\kappa|_{H^{\frac{3}{2}k-1}(S)}^2 \leq C_0 E, \quad |v|_{H^{\frac{3}{2}k}(\mathbb{R}^n \setminus S)}^2 \leq C_0 (E + E_0)^m$$

for some integer $m > 0$ depending only on k and n and some constant $C_0 > 0$ depending only on the set Λ_0 .

The proof of this proposition will be given below. Using this result we will prove the following theorem on energy estimates.

Theorem 4.1. Fix $\delta > 0$ sufficiently small. Then there exists $L > 0$ such that, if a solution of (E) and (BC) is given by S_t with $S_t \in H^{\frac{3}{2}k+1}$ and $v(t, \cdot) \in C_t^0(H^{\frac{3}{2}k}(\mathbb{R}^n \setminus S_t))$, then there exists $t^* > 0$,

depending only on $|v(0, \cdot)|_{H^{\frac{3}{2}k}(\mathbb{R}^n \setminus S_t)}$, L , and the set Λ_0 , such that, for all $t \in [0, t^*]$,

$$(4.4) \quad \begin{aligned} S_t \in \Lambda_0 \quad \text{and} \quad |\kappa|_{H^{\frac{3}{2}k-1}(S_t)} &\leq L, \\ E(S_t, v(t, \cdot)) &\leq 2E(S_0, v(0, \cdot)) + C_1 + \int_0^t P(E_0, E(S_{t'}, v(t', \cdot))) dt' \end{aligned}$$

where $P(\cdot)$ is a polynomial of positive coefficients determined only by the set Λ_0 and C_1 is a constant determined only by $|v(0, \cdot)|_{H^{\frac{3}{2}k-\frac{3}{2}}(\mathbb{R}^n \setminus S_0)}$, and the set Λ_0 .

Since the domain is evolving, the above continuity assumption of v in t means that there exist extensions of both v_+ and v_- to $[0, T] \times \mathbb{R}^n$ which are continuous in $H^{\frac{3}{2}k}(\mathbb{R}^n)$.

In order to prove Proposition 4.1 and Theorem 4.1, we need the following lemmas.

Lemma 4.2. *For any $S \in \Lambda_0$ with $\kappa \in H^s(S)$, $s \in [\frac{3}{2}k - \frac{5}{2}, \frac{3}{2}k - 1]$, we have*

$$|\Pi|_{H^s(S)} + |N|_{H^{s+1}(S)} \leq C(1 + |\kappa|_{H^s(S)}),$$

for some $C > 0$ uniform in $S \in \Lambda_0$.

Proof. The proof is a straightforward application of elliptic regularity applied to

$$(4.5) \quad -\Delta_S \Pi = -\mathcal{D}^2 \kappa + (|\Pi|^2 I - \kappa \Pi) \Pi.$$

and the fact that $|\Pi|_{H^{\frac{n}{2}-1}(S)} \leq C$ uniform in $S \in \Lambda_0$. \square

Corollary. *Suppose $S \in \Lambda_0$ with $\kappa \in H^{\frac{3}{2}k-\frac{3}{2}}(S)$, $g \in H^{\frac{3}{2}k-1}(\Omega^\pm)$, and $q = -\Delta_\pm^{-1}g$, then we have*

$$|\nabla_{N_{\mathcal{H}_\pm}} q|_{H^{\frac{3}{2}k}(\Omega^\pm)} \leq C(1 + |\kappa|_{H^{\frac{3}{2}k-\frac{3}{2}}(S)})|g|_{H^{\frac{3}{2}k-1}(\Omega^\pm)}$$

for some $C > 0$ uniform in $S \in \Lambda_0$.

The proof of this corollary follows Lemma 4.2 and the identities

$$\begin{aligned} \nabla_{N_\pm} \nabla_{N_{\mathcal{H}_\pm}} q &= \mathcal{N}(N_\pm) \cdot \nabla q + D^2 q(N_\pm, N_\pm) = \mathcal{N}(N_\pm) \cdot \nabla q - g - \kappa_\pm \nabla_{N_\pm} q && \text{on } S \\ -\Delta \nabla_{N_{\mathcal{H}_\pm}} q &= \nabla_{N_{\mathcal{H}_\pm}} g - 2D^2 q \cdot DN_{\mathcal{H}_\pm} && \text{in } \Omega^\pm. \end{aligned}$$

Lemma 4.3. *Suppose $S \in \Lambda_0$ with $\kappa \in H^{\frac{3}{2}k-\frac{3}{2}}(S)$,*

$$|(-\Delta_S)^{\frac{1}{2}} - \mathcal{N}_\pm|_{L(H^{s'}(S))} \leq C(1 + |\kappa|_{H^{\frac{3}{2}k-\frac{3}{2}}(S)}), \quad s' \in [\frac{1}{2} - \frac{3}{2}k, \frac{3}{2}k - \frac{1}{2}].$$

Proof. From the identity

$$(-\Delta_S - \mathcal{N}_\pm^2)f = \kappa_\pm \mathcal{N}_\pm(f) - 2\nabla_{N_\pm}(-\Delta_\pm)^{-1}(DN_{\mathcal{H}_\pm} \cdot D^2 f_{\mathcal{H}_\pm}) - \mathcal{N}_\pm(N_\pm) \cdot (\mathcal{N}_\pm(f)N_\pm + \nabla^\top f)$$

for any smooth $f : S \rightarrow \mathbb{R}$, and lemma 4.2 we have

$$|-\Delta_S - \mathcal{N}_\pm^2|_{L(H^{s'}(S), H^{s'-1}(S))} \leq C$$

Using commutators estimates $[-\Delta_S, \mathcal{N}_\pm]$ and factorization, the lemma follows. A detailed proof of the commutators and the factorization is given in section 6 of [SZ06]. \square

Proof of Proposition 4.1. The two terms $|\mathcal{A}^k v|_{L^2(S)}^2$ and $|\mathcal{A}^{k-\frac{1}{2}} \nabla p_\kappa|_{L^2(S)}^2$ can be written explicitly using the definition (3.21) of \mathcal{A}

$$(4.6) \quad |\mathcal{A}^{\frac{k}{2}} v|_{L^2(\mathbb{R}^n \setminus S)}^2 = \int_S v_+^\perp (-\Delta_S \bar{\mathcal{N}})^{k-1} (-\Delta_S) v_+^\perp dS$$

$$(4.7) \quad |\mathcal{A}^{\frac{k}{2}-\frac{1}{2}} \nabla p_\kappa|_{L^2(\mathbb{R}^n \setminus S)}^2 = \int_S \kappa_+ \bar{\mathcal{N}} (-\Delta_S \bar{\mathcal{N}})^{k-1} \kappa_+ dS,$$

where

$$(4.8) \quad \bar{\mathcal{N}} = \left(\frac{1}{\rho_+} \mathcal{N}_+\right) \mathcal{N}^{-1} \left(\frac{1}{\rho_-} \mathcal{N}_-\right) = \left(\left(\frac{1}{\rho_+} \mathcal{N}_+\right)^{-1} + \left(\frac{1}{\rho_-} \mathcal{N}_-\right)^{-1}\right)^{-1}.$$

Clearly $\bar{\mathcal{N}}$ is also self-adjoint and positive. The estimates on $|\kappa|_{H^{\frac{3}{2}k-1}(S)}$ and $|v_+^\perp|_{H^{\frac{3}{2}k-\frac{1}{2}}(S)}$ follow immediately since from lemma 4.3 \mathcal{N}_\pm behaves like $(-\Delta_S)^{\frac{1}{2}}$.

To bound v in terms of E_0 and E we note that Δv is controlled by E from

$$(4.9) \quad \Delta v^i = \partial_j \omega_j^i.$$

Therefore it is sufficient to control the boundary value $\nabla_{N_\pm} v_\pm$ by E . Moreover from the identity $\nabla_{N_\pm} v_\pm = (Dv_\pm)^*(N_\pm) + \omega_\pm(N_\pm)$ where ω_\pm is the restriction of ω on S , it suffices to show that E_0 and E control $\nu_\pm = (Dv_\pm)^*(N_\pm)$.

We first estimate ν_+^\top using the identity

$$(4.10) \quad \Delta_S \nu_+^\top = \nabla^\top (\mathcal{D} \cdot \nu_+^\top) + \text{Ric}((Dv_+)^*(N_+)^\top) + (\mathcal{D}_{X_j} \omega_{\nu_+}^\top)(X_j), \quad \text{at } x \in S$$

where Ric is the Ricci curvature of S , $\{X_1, \dots, X_{n-1}\}$ is any orthonormal frame of $T_x S$, and the tangential curl $\omega_{\nu_+}^\top$ of ν_+^\top is defined as

$$\omega_{\nu_+}^\top(x)(X) \cdot Y = \mathcal{D}_X \nu_+^\top \cdot Y - \mathcal{D}_Y \nu_+^\top \cdot X = \nabla_X \nu_+^\top \cdot Y - \nabla_Y \nu_+^\top \cdot X$$

for any $X, Y \in T_x S$. From the definition of ν_\pm we have

$$\omega_{\nu_+}^\top(X) \cdot Y = \Pi_+(X) \cdot \nabla_Y v_+ - \Pi_+(Y) \cdot \nabla_X v_+$$

Therefore, by Sobolev inequalities, there exists $C > 0$ uniform in $S \in \Lambda_0$ so that

$$\begin{aligned} |\Delta_S \nu_+^\top|_{H^{\frac{3k-7}{2}}(S)} &\leq |\mathcal{D} \cdot \nu_+^\top|_{H^{\frac{3k}{2}-\frac{5}{2}}(S)} + |\text{Ric}|_{H^{\frac{3k}{2}-\frac{5}{2}}(S)} |N_+|_{H^{\frac{3k}{2}-\frac{3}{2}}(S)} |Dv_+|_{H^{\frac{3k}{2}-2}(S)} + |\omega_{\nu_+}^\top|_{H^{\frac{3k}{2}-\frac{5}{2}}(S)} \\ &\leq |\mathcal{D} \cdot \nu_+^\top|_{H^{\frac{3}{2}k-\frac{5}{2}}(S)} + C|v_+|_{H^{\frac{3}{2}k-\frac{1}{8}}(\Omega^+)}. \end{aligned}$$

Here the norm $H^{\frac{3}{2}k-\frac{1}{8}}$ is chosen to illustrate that the term is lower order. In fact any $H^{\frac{3}{2}k-\alpha}$ with $0 < \alpha < \frac{1}{2}$ works. To estimate the divergence term $\mathcal{D} \cdot \nu_+^\top$, one may compute

$$\mathcal{D} \cdot \nu_+^\top = \Delta_S v_+^\perp - \mathcal{D} \cdot (\Pi_+(v_+^\top)),$$

which along with Lemma 4.2 implies

$$\begin{aligned} |\mathcal{D} \cdot \nu_+^\top|_{H^{\frac{3k}{2}-\frac{5}{2}}(S)} &\leq C|v_+^\perp|_{H^{\frac{3}{2}k-\frac{1}{2}}(S)} + C|\Pi_+|_{H^{\frac{3}{2}k-\frac{3}{2}}(S)} |v_+ - (v_+ \cdot N_+)N_+|_{H^{\frac{3}{2}k-\frac{3}{2}}(S)} \\ &\leq C|v_+^\perp|_{H^{\frac{3}{2}k-\frac{1}{2}}(S)} + C(1 + |\kappa_+|_{H^{\frac{3}{2}k-\frac{3}{2}}(S)}) |v_+|_{H^{\frac{3}{2}k-1}(\Omega^+)}. \end{aligned}$$

Therefore, from (4.6) and (4.7), we obtain

$$(4.11) \quad |\Delta_S \nu_+^\top|_{H^{\frac{3}{2}k-\frac{7}{2}}(S)} \leq CE^{\frac{1}{2}} + C(1 + E^{\frac{1}{3}}) |v_+|_{H^{\frac{3}{2}k-\frac{1}{8}}(\Omega^+)}.$$

Finally, we only need to estimate $\nu_+^\perp = \nabla_{N_+} v_+ \cdot N_+$, much as in the way in [SZ06]. Extending ν_+ into Ω^+ as $\nu_+ = (Dv_+)^*(N_{\mathcal{H}_+})$, where $N_{\mathcal{H}_+}$ is the harmonic extension of N_+ into Ω^+ , and comparing the two ways of computing $\nabla \cdot \nu_+$ on S using 1) frames and ω and 2) divergence decomposition formula on S , we obtain

$$\nabla_{N_+} \nu_+ \cdot N_+ = (\nabla_{X_i} \omega)(X_i) \cdot N_+ + Dv_+ \cdot DN_{\mathcal{H}_+} - \mathcal{D} \cdot \nu_+^\top - \kappa_+ \nu_+^\perp.$$

Moreover,

$$\begin{aligned} \nabla_{N_+} \nu_+ \cdot N_+ &= \nabla_{N_+} (\nabla_{N_{\mathcal{H}_+}} v_+ \cdot N_{\mathcal{H}_+}) - N_+ \cdot Dv_+ (\mathcal{N}_+(N_{\mathcal{H}_+})) \\ &= \mathcal{N}_+(\nu_+^\perp) + \nabla_{N_+} \Delta_+^{-1} \Delta (\nabla_{N_{\mathcal{H}_+}} v_+ \cdot N_{\mathcal{H}_+}) - N_+ \cdot Dv_+ (\mathcal{N}_+(N_{\mathcal{H}_+})). \end{aligned}$$

Therefore, from the estimate on $\mathcal{D} \cdot \nu_+^\top$, we obtain

$$|\mathcal{N}_+ \nu_+^\perp|_{H^{\frac{3k}{2}-\frac{5}{2}}(S)} \leq CE^{\frac{1}{2}} + C(1 + E^{\frac{1}{3}})|v_+|_{H^{\frac{3k}{2}-\frac{1}{8}}(\Omega^+)},$$

which, along with (4.9) and (4.11), implies

$$|v_+|^2_{H^{\frac{3k}{2}}(\Omega^+)} \leq CE + C(1 + E^{\frac{2}{3}})|v_+|^2_{H^{\frac{3k}{2}-\frac{1}{8}}(\Omega^+)}.$$

The estimate in Proposition 4.1 follows immediately from Sobolev inequalities. \square

Proof of Theorem 4.1. To prove Theorem 4.1, in addition to Proposition 4.1, we need the following: a) the estimates on the Lagrangian coordinates map and consequently $\kappa \in H^{\frac{3k}{2}-\frac{5}{2}}(S)$, b) estimates on $\omega = Dv - (Dv)^*$, and c) commutators involving \mathbf{D}_t . In the following all constant $C > 0$ will depend only on the set Λ_0 .

Estimate of the Lagrangian coordinate map $u(t, y)$. We will only work on the domain Ω_t^t . From our assumption on v , the ODE $u_t(t, y) = v(t, u(t, y))$ solving u is well-posed. Since $u(t, \cdot) : \Omega_0^+ \rightarrow \Omega_t^+$ is volume preserving and $\frac{3}{2}k > \frac{n}{2} + 1$, it is easy to derive, Therefore,

$$(4.12) \quad |u(t, \cdot) - I|_{H^{\frac{3k}{2}}(\Omega_0^+)} \leq C \int_0^t |v(t', \cdot)|_{H^{\frac{3k}{2}}(\Omega_{t'}^+)} |u(t', \cdot)|_{H^{\frac{3k}{2}}(\Omega_{t'}^+)}^{\frac{3}{2}k} dt',$$

where $C > 0$ depends only on n and k . Let $\mu > 0$ be a positive large number specified later,

$$(4.13) \quad t_0 = \sup\{t \mid |v(t', \cdot)|_{H^{\frac{3k}{2}}(\mathbb{R}^n \setminus S_t)} < \mu, \forall t' \in [0, t]\},$$

We have $t_0 > 0$ due to the continuity of $v(t, \cdot)$ in $H^{\frac{3k}{2}}(\Omega_t)$. From ODE estimates, there exists $t_1 > 0$ and $C_2 > 0$ which depend only on μ such that, for all $0 \leq t \leq \min\{t_0, t_1\}$,

$$(4.14) \quad |u(t, \cdot) - I|_{H^{\frac{3k}{2}}(\Omega_0^+)} \leq C_2 t.$$

It implies the mean curvature estimate, for all $0 \leq t \leq \min\{t_0, t_1\}$,

$$(4.15) \quad |\kappa(t, \cdot)|_{H^{\frac{3k}{2}-\frac{5}{2}}(S_t)} \leq |\kappa(0, \cdot)|_{H^{\frac{3k}{2}-\frac{5}{2}}(S_0)} + C_3 t.$$

Here it is easy to see from local coordinates that C_3 is determined only by μ and the set Λ_0 . Therefore, there exists $t_2 > 0$ determined only by μ and the set Λ_0 such that $S_t \in \Lambda_0$ for $0 \leq t \leq \min\{t_0, t_2\}$.

Evolution of the curl $\omega = Dv - (Dv)^$.* It is easy to compute

$$\mathbf{D}_t \omega = D\mathbf{D}_t v - (D\mathbf{D}_t v)^* + ((Dv)^*)^2 - (Dv)^2 = ((Dv)^*)^2 - (Dv)^2 = -(Dv)^* \omega - \omega Dv.$$

Since $\frac{3}{2}k > \frac{n}{2} + 1$, we have

$$(4.16) \quad \frac{d}{dt} \int_{\mathbb{R}^n \setminus S_t} |D^{\frac{3}{2}k-1} \omega|_{H^{\frac{3}{2}k-1}(\mathbb{R}^n \setminus S_t)}^2 dx \leq C |v|_{H^{\frac{3}{2}k}(\mathbb{R}^n \setminus S_t)} |\omega|_{H^{\frac{3}{2}k-1}(\mathbb{R}^n \setminus S_t)}^2.$$

The commutator involving \mathbf{D}_t . First we list a few commutators calculated in [SZ06]:

$$(4.17) \quad \mathbf{D}_t \nabla g = \nabla \mathbf{D}_t g - (Dv)^*(\nabla g),$$

$$(4.18) \quad [\mathbf{D}_t, \Delta_\pm^{-1}]g = \Delta_\pm^{-1}(2Dv \cdot D^2 \Delta_\pm^{-1} g + \Delta v \cdot \nabla \Delta_\pm^{-1} g),$$

for any smooth function g . It also has been proved in [SZ06] that, for any function f defined on S_t ,

$$(4.19) \quad |[\mathbf{D}_t, \Delta_{S_t}]|_{L(H^{s_1}(S_t), H^{s_1-2}(S_t))} \leq C |v|_{H^{\frac{3}{2}k}(\mathbb{R}^n \setminus S_t)} \quad s_1 \in \left(\frac{7}{2} - \frac{3}{2}k, \frac{3}{2}k - \frac{1}{2}\right],$$

$$(4.20) \quad |[\mathbf{D}_t, \mathcal{N}_\pm]|_{L(H^{s_2}(S_t), H^{s_2-1}(S_t))} \leq C |v|_{H^{\frac{3}{2}k}(\mathbb{R}^n \setminus S_t)} \quad s_2 \in \left[\frac{1}{2}, \frac{3}{2}k - \frac{1}{2}\right].$$

It also implies that

$$(4.21) \quad \|[\mathbf{D}_t, \mathcal{N}^{-1}]\|_{L(H^s(S_t), H^{s-1}(S_t))} \leq C|v|_{H^{\frac{3}{2}k}(\mathbb{R}^n \setminus S_t)} \quad s \in [-\frac{1}{2}, \frac{3}{2}k - \frac{3}{2}]$$

$$(4.22) \quad \|[\mathbf{D}_t, \bar{\mathcal{N}}]\|_{L(H^s(S_t), H^{s-1}(S_t))} \leq C|v|_{H^{\frac{3}{2}k}(\mathbb{R}^n \setminus S_t)} \quad s \in [\frac{1}{2}, \frac{3}{2}k - \frac{1}{2}]$$

Evolution of E. For the rest of this section, let $Q = Q(|v|_{H^{\frac{3}{2}k}(\mathbb{R}^n \setminus S_t)}, |\kappa|_{H^{\frac{3}{2}k-1}(S_t)})$ denote a generic positive polynomial in $|v|_{H^{\frac{3}{2}k}(\mathbb{R}^n \setminus S_t)}$, and $|\kappa|_{H^{\frac{3}{2}k-1}(S_t)}$ with coefficients depending only on the set Λ_0 . One may notice here, by

Lemma 4.2, $|\Pi|_{H^{\frac{3}{2}k-1}(S_t)}$, and $|v_\pm^\perp|_{H^{\frac{3}{2}k-\frac{1}{2}}(S_t)}$, and $|v_\pm^\top|_{H^{\frac{3}{2}k-\frac{1}{2}}(S_t)}$ can also be included in Q . Recall that

$$\mathbf{D}_{t\pm} dS = (\mathcal{D} \cdot v_\pm^\top + \kappa_\pm v_\pm^\perp) dS.$$

Since

$$\nabla \cdot v_\pm|_{S_t} = \mathcal{D} \cdot v_\pm^\top + \kappa_\pm v_\pm^\perp + \nabla_{N_\pm} v_\pm \cdot N_\pm = 0$$

then

$$|\kappa_\pm v_\pm^\perp + \mathcal{D} \cdot v_\pm^\top|_{H^{\frac{3}{2}k-\frac{3}{2}}(S_t)} = |\nabla_{N_\pm} v_\pm \cdot N_\pm|_{H^{\frac{3}{2}k-\frac{3}{2}}(S_t)} \leq C|v|_{H^{\frac{3}{2}k}(\mathbb{R}^n \setminus S_t)}$$

and thus $\mathbf{D}_t dS$ would not complicate the estimates since $\frac{3}{2}k \geq \frac{n}{2} + \frac{3}{2}$.

Before we embark on calculating the energy inequality it is helpful to recall two facts. First, we only need to keep track of terms which can not be bounded by $|v_\pm|_{H^{\frac{3}{2}k}(\Omega_t^\pm)}$ and $|\kappa|_{H^{\frac{3}{2}k-1}(S_t)}$, and second, \mathcal{N}_\pm are selfadjoint operators of order one.

$$\mathbf{I}: \quad \left| \frac{1}{2} \frac{d}{dt} |\mathcal{A}^{\frac{k}{2}-\frac{1}{2}} \nabla p_\kappa|_{L^2(\mathbb{R}^n \setminus S_t)}^2 - \int_{S_t} \kappa_+ \bar{\mathcal{N}} (-\Delta_{S_t} \bar{\mathcal{N}})^{k-1} (-\Delta_{S_t}) v_+^\perp dS \right| \leq Q.$$

From (4.7), (4.19), and (4.22), it is clear

$$\left| \frac{1}{2} \frac{d}{dt} |\mathcal{A}^{\frac{k}{2}-\frac{1}{2}} \nabla p_\kappa|_{L^2(\mathbb{R}^n \setminus S_t)}^2 - \int_{S_t} \kappa_+ \bar{\mathcal{N}} (-\Delta_{S_t} \bar{\mathcal{N}})^{k-1} \mathbf{D}_{t+} \kappa_+ dS \right| \leq Q.$$

From the expression for $\mathbf{D}_{t\pm} \kappa_\pm$ given in [SZ06]

$$(4.23) \quad \mathbf{D}_{t\pm} \kappa_\pm = -\Delta_{S_t} v_\pm^\perp - v_\pm^\perp |\Pi|^2 + (\mathcal{D} \cdot \Pi_\pm)(v_\pm^\top) = -\Delta_{S_t} v_\pm^\perp - v_\pm^\perp |\Pi|^2 + \nabla_{v_\pm^\top} \kappa_\pm$$

and Lemma 4.2 we only need

$$(4.24) \quad \left| \int_{S_t} \kappa_+ \bar{\mathcal{N}} (-\Delta_{S_t} \bar{\mathcal{N}})^{k-1} \nabla_{v_+^\top} \kappa_+ dS \right| \leq Q.$$

to derive estimate **I**. By considering a flow $\phi(\tau, \cdot)$ on Ω_t^+ generated by $\mathcal{H}_+ v_+^\top$, the above commutator estimates applied to \mathbf{D}_τ allow us to pull $\nabla_{v_+^\top}$ to the front with lower order errors

$$\left| \int_{S_t} \kappa_+ \bar{\mathcal{N}} (-\Delta_{S_t} \bar{\mathcal{N}})^{k-1} \nabla_{v_+^\top} \kappa_+ dS - \frac{1}{2} \int_{S_t} \nabla_{v_+^\top} \left(\kappa_+ \bar{\mathcal{N}} (-\Delta_{S_t} \bar{\mathcal{N}})^{k-1} \kappa_+ \right) dS \right| \leq Q.$$

Therefore, inequality (4.24) follows from the Divergence Theorem and **I** follows consequently.

$$\mathbf{II}: \quad \left| \frac{d}{dt} \left(\frac{1}{2} |\mathcal{A}^{\frac{k}{2}} v|_{L^2(\mathbb{R}^n \setminus S_t)}^2 - E_{ex} \right) + \int_{S_t} v_+^\perp (-\Delta_{S_t} \bar{\mathcal{N}})^k \kappa_+ dS \right| \leq Q$$

where the extra term

$$\begin{aligned} E_{ex} = & \frac{\rho_+}{2(\rho_+ + \rho_-)} \int_{S_t} \nabla_{v_+^\top} \kappa_+ \cdot \bar{\mathcal{N}} (-\Delta_{S_t} \bar{\mathcal{N}})^{k-2} \nabla_{v_+^\top} \kappa_+ dS \\ & - \frac{\rho_-}{2(\rho_+ + \rho_-)} \int_{S_t} \nabla_{v_-^\top} \kappa_- \cdot \bar{\mathcal{N}} (-\Delta_{S_t} \bar{\mathcal{N}})^{k-2} \nabla_{v_-^\top} \kappa_- dS \}. \end{aligned}$$

From (4.6), (4.19), and (4.22), it is clear

$$(4.25) \quad \left| \frac{1}{2} \frac{d}{dt} |\mathcal{A}^{\frac{k}{2}} v|_{L^2(\mathbb{R}^n \setminus S_t)}^2 - \int_{S_t} v_+^\perp (-\Delta_{S_t} \bar{\mathcal{N}})^{k-1} (-\Delta_{S_t}) \mathbf{D}_{t+} v_+^\perp dS \right| \leq Q.$$

Using (3.18), (2.1), (3.17), and (3.11), we have

$$\begin{aligned} \mathbf{D}_{t+} v_+^\perp &= (\mathbf{D}_{t+} v_+) \cdot N_+ + v_+ \cdot \mathbf{D}_{t+} N_+ = -\nabla_{N_+} p_{v,v}^+ - \nabla_{N_+} p_\kappa^+ - \nabla_{v_+^\perp} v_+ \cdot N_+ \\ &= -\frac{1}{\rho_+} \mathcal{N}_+ p_{v,v}^S - \nabla_{N_+} \Delta_+^{-1} \text{tr}(Dv)^2 - \bar{\mathcal{N}} \kappa_+ - \nabla_{v_+^\perp} v_+^\perp + \Pi_+(v_+^\top, v_+^\top). \end{aligned}$$

From the corollary of Lemma 4.2, we have

$$(4.26) \quad |\nabla_{N_\pm} \Delta_\pm^{-1} \text{tr}(Dv)^2|_{H^{\frac{3}{2}k-\frac{1}{2}}(S_t)} \leq Q.$$

Therefore,

$$(4.27) \quad \left| \frac{1}{2} \frac{d}{dt} |\mathcal{A}^{\frac{k}{2}} v|_{L^2(\mathbb{R}^n \setminus S_t)}^2 - \int_{S_t} v_+^\perp (-\Delta_{S_t} \bar{\mathcal{N}})^{k-1} (-\Delta_{S_t}) \left(-\frac{1}{\rho_+} \mathcal{N}_+ p_{v,v}^S - \bar{\mathcal{N}} \kappa_+ - \nabla_{v_+^\perp} v_+^\perp + \Pi_+(v_+^\top, v_+^\top) \right) dS \right| \leq Q.$$

From (3.11), we can write the first term in the second integral as

$$\begin{aligned} -\frac{1}{\rho_+} \mathcal{N}_+ p_{v,v}^S &= \frac{1}{\rho_+} \mathcal{N}_+ \bar{\mathcal{N}}^{-1} \{ 2 \nabla_{v_+^\top - v_-^\top} v_+^\perp - \Pi_+(v_+^\top, v_+^\top) - \Pi_-(v_-^\top, v_-^\top) \\ &\quad - \nabla_{N_+} \Delta_+^{-1} \text{tr}(Dv)^2 - \nabla_{N_-} \Delta_-^{-1} \text{tr}(Dv)^2 \} \end{aligned}$$

Since $|\bar{\mathcal{N}}^{-1} \mathcal{N}_+|_{L(H^{\frac{3}{2}k-\frac{1}{2}}(S_t))} \leq C$, we have that its dual $\mathcal{N}_+ \bar{\mathcal{N}}^{-1}$ satisfies the same estimate. Therefore, using (4.26), we obtain

$$\left| -\frac{1}{\rho_+} \mathcal{N}_+ p_{v,v}^S - \frac{1}{\rho_+} \mathcal{N}_+ \bar{\mathcal{N}}^{-1} (2 \nabla_{v_+^\top - v_-^\top} v_+^\perp - \Pi_+(v_+^\top, v_+^\top) - \Pi_-(v_-^\top, v_-^\top)) \right|_{H^{\frac{3}{2}k-\frac{1}{2}}(S_t)} \leq Q.$$

From Lemma 4.3, we have

$$\left| \mathcal{N}_+ \bar{\mathcal{N}}^{-1} - \frac{\rho_+ \rho_-}{\rho_+ + \rho_-} \right|_{L(H^{\frac{3}{2}k-\frac{3}{2}}(S_t), H^{\frac{3}{2}k-\frac{1}{2}}(S_t))} \leq Q,$$

which, along with Lemma 4.2, implies

$$\left| -\frac{1}{\rho_+} \mathcal{N}_+ p_{v,v}^S - \frac{\rho_-}{\rho_+ + \rho_-} (2 \nabla_{v_+^\top - v_-^\top} v_+^\perp - \Pi_+(v_+^\top, v_+^\top) - \Pi_-(v_-^\top, v_-^\top)) \right|_{H^{\frac{3}{2}k-\frac{1}{2}}(S_t)} \leq Q.$$

Substituting this estimate into (4.27), we have

$$\begin{aligned} \left| \frac{1}{2} \frac{d}{dt} |\mathcal{A}^{\frac{k}{2}} v|_{L^2(\mathbb{R}^n \setminus S_t)}^2 - \int_{S_t} v_+^\perp (-\Delta_{S_t} \bar{\mathcal{N}})^{k-1} (-\Delta_{S_t}) \left\{ \frac{\rho_+}{\rho_+ + \rho_-} \Pi_+(v_+^\top, v_+^\top) - \frac{\rho_-}{\rho_+ + \rho_-} \Pi_-(v_-^\top, v_-^\top) \right. \right. \\ \left. \left. - \bar{\mathcal{N}} \kappa_+ + \nabla v_+^\perp \cdot \left(\frac{\rho_- - \rho_+}{\rho_+ + \rho_-} v_+^\top - \frac{2\rho_-}{\rho_+ + \rho_-} v_-^\top \right) \right\} dS \right| \leq Q. \end{aligned}$$

Much as the proof of inequality (4.24), by commuting $\nabla_{v_\pm^\top}$ we obtain

$$\left| \int_{S_t} v_+^\perp (-\Delta_{S_t} \bar{\mathcal{N}})^{k-1} (-\Delta_{S_t}) \nabla v_+^\perp \cdot \left(\frac{\rho_- - \rho_+}{\rho_+ + \rho_-} v_+^\top - \frac{2\rho_-}{\rho_+ + \rho_-} v_-^\top \right) dS \right| \leq Q.$$

Therefore, we have

$$(4.28) \quad \left| \frac{1}{2} \frac{d}{dt} |\mathcal{A}^{\frac{k}{2}} v|_{L^2(\mathbb{R}^n \setminus S_t)}^2 - \int_{S_t} v_+^\perp (-\Delta_{S_t} \bar{\mathcal{N}})^{k-1} (-\Delta_{S_t}) \left\{ \frac{\rho_+}{\rho_+ + \rho_-} \Pi_+(v_+^\top, v_+^\top) - \frac{\rho_-}{\rho_+ + \rho_-} \Pi_-(v_-^\top, v_-^\top) - \bar{\mathcal{N}} \kappa_+ \right\} dS \right| \leq Q.$$

In order to estimate the terms with Π , we first use Lemma 4.2 and identity (4.5) to obtain

$$|\Delta_{S_t}(\Pi_{\pm}(v_{\pm}^{\top}, v_{\pm}^{\top})) - \mathcal{D}^2 \kappa_{\pm}(v_{\pm}^{\top}, v_{\pm}^{\top})|_{H^{\frac{3}{2}k-\frac{5}{2}}(S_t)} \leq Q.$$

Since

$$\mathcal{D}^2 \kappa_{\pm}(v_{\pm}^{\top}, v_{\pm}^{\top}) = \nabla_{v_{\pm}^{\top}} \nabla_{v_{\pm}^{\top}} \kappa_{\pm} - \mathcal{D}_{v_{\pm}^{\top}} v_{\pm}^{\top} \cdot \nabla \kappa_{\pm}$$

we have

$$\begin{aligned} \left| \frac{1}{2} \frac{d}{dt} |\mathcal{A}^{\frac{k}{2}} v|_{L^2(\mathbb{R}^n \setminus S_t)}^2 - \int_{S_t} v_{+}^{\perp} (-\Delta_{S_t} \bar{\mathcal{N}})^{k-1} \left\{ \Delta_{S_t} \bar{\mathcal{N}} \kappa_{+} - \frac{\rho_{+}}{\rho_{+} + \rho_{-}} \nabla_{v_{+}^{\top}} \nabla_{v_{+}^{\top}} \kappa_{+} \right. \right. \\ \left. \left. + \frac{\rho_{-}}{\rho_{+} + \rho_{-}} \nabla_{v_{-}^{\top}} \nabla_{v_{-}^{\top}} \kappa_{-} \right\} dS \right| \leq Q. \end{aligned}$$

As in the proof of (4.24), we apply the commutator estimates (4.19) and (4.20) to move one of $\nabla_{v_{\pm}^{\top}}$ and obtain

$$\begin{aligned} \left| \frac{1}{2} \frac{d}{dt} |\mathcal{A}^{\frac{k}{2}} v|_{L^2(\mathbb{R}^n \setminus S_t)}^2 - \frac{\rho_{+}}{\rho_{+} + \rho_{-}} \int_{S_t} \nabla_{v_{+}^{\top}} (-\Delta_{S_t} v_{+}^{\perp}) \cdot \bar{\mathcal{N}} (-\Delta_{S_t} \bar{\mathcal{N}})^{k-2} \nabla_{v_{+}^{\top}} \kappa_{+} dS \right. \\ \left. - \frac{\rho_{-}}{\rho_{+} + \rho_{-}} \int_{S_t} \nabla_{v_{-}^{\top}} (-\Delta_{S_t} v_{-}^{\perp}) \cdot \bar{\mathcal{N}} (-\Delta_{S_t} \bar{\mathcal{N}})^{k-2} \nabla_{v_{-}^{\top}} \kappa_{-} dS + \int_{S_t} v_{+}^{\perp} (-\Delta_{S_t} \bar{\mathcal{N}})^k \kappa_{+} dS \right| \leq Q. \end{aligned}$$

From identity (4.23),

$$|-\Delta_{S_t} v_{+}^{\perp} - \mathbf{D}_{t_{+}} \kappa_{+}|_{H^{\frac{3}{2}k-2}(S_t)} \leq Q,$$

which implies

$$\begin{aligned} \left| \frac{1}{2} \frac{d}{dt} |\mathcal{A}^{\frac{k}{2}} v|_{L^2(\mathbb{R}^n \setminus S_t)}^2 - \frac{\rho_{+}}{\rho_{+} + \rho_{-}} \int_{S_t} \nabla_{v_{+}^{\top}} \mathbf{D}_{t_{+}} \kappa_{+} \cdot \bar{\mathcal{N}} (-\Delta_{S_t} \bar{\mathcal{N}})^{k-2} \nabla_{v_{+}^{\top}} \kappa_{+} dS \right. \\ \left. - \frac{\rho_{-}}{\rho_{+} + \rho_{-}} \int_{S_t} \nabla_{v_{-}^{\top}} \mathbf{D}_{t_{-}} \kappa_{-} \cdot \bar{\mathcal{N}} (-\Delta_{S_t} \bar{\mathcal{N}})^{k-2} \nabla_{v_{-}^{\top}} \kappa_{-} dS + \int_{S_t} v_{+}^{\perp} (-\Delta_{S_t} \bar{\mathcal{N}})^k \kappa_{+} dS \right| \leq Q. \end{aligned}$$

Using (4.19) and (4.20) one more time, we obtain inequality **II**.

Proof of Theorem 4.1. Adding inequalities (4.16), **I**, and **II**, we have

$$(4.29) \quad E(t) - E(0) + E_{ex}(t) - E_{ex}(0) \leq \int_0^t Q(|v|_{H^{3k}(\mathbb{R}^n \setminus S_{t'})}, |\kappa|_{H^{3k-1}(S_{t'})}) dt',$$

where Q is a polynomial with positive coefficients that depends only on Λ_0 . This inequality holds on $[0, \min\{t_0, t_2\}]$ where t_0 is defined in (4.13) and t_2 is determined only by $|v(0, \cdot)|_{H^{3k}(\mathbb{R}^n \setminus S_t)}$ and the set Λ_0 , the neighborhood of S_0 in $H^{\frac{3}{2}k-\frac{1}{2}}$. Clearly

$$|E_{ex}| \leq C |v|_{H^{\frac{3}{2}k-\frac{5}{8}}(\mathbb{R}^n \setminus S_t)}^2 |\kappa|_{H^{\frac{3}{2}k-\frac{3}{2}}(S_t)}^2,$$

where $C > 0$ depends only on the set Λ_0 . Interpolating v between $H^{3k}(\mathbb{R}^n \setminus S_t)$ and $H^{3k-\frac{3}{2}}(\mathbb{R}^n \setminus S_t)$ and κ between $H^{3k-1}(S_t)$ and $H^{3k-\frac{5}{2}}(S_t)$, we obtain from Proposition 4.1,

$$|E_{ex}| \leq \frac{1}{2} E + C_1 (1 + |v|_{H^{3k-\frac{3}{2}}(\mathbb{R}^n \setminus S_t)}^m)$$

for some integer $m > 0$ where the constant C_1 , which include $|\kappa|_{H^{3k-\frac{5}{2}}(S_t)}$, is determined only by E_0 and the set Λ_0 . From the Euler's equation (3.18), (4.2), (3.11), Lemma 4.2, and its corollary,

$$|\mathbf{D}_t v|_{H^{\frac{3}{2}k-\frac{3}{2}}(\mathbb{R}^n \setminus S_t)} = |\nabla p_{v,v} + \nabla p_{\kappa}|_{H^{\frac{3}{2}k-\frac{3}{2}}(\mathbb{R}^n \setminus S_t)} \leq Q.$$

We can use the Lagrangian coordinate map $u(t, \cdot)$ to estimate

$$|v(t, \cdot)|_{H^{3k-\frac{3}{2}}(\mathbb{R}^n \setminus S_t)} - |v(0, \cdot)|_{H^{3k-\frac{3}{2}}(\mathbb{R}^n \setminus S_0)}$$

Through a similar procedure of the derivation of (4.14), there exists $t_3 > 0$, depending only on $|v(0, \cdot)|_{H^{3k}(\mathbb{R}^n \setminus S_t)}$ and the set Λ_0 so that for $0 \leq t \leq \min\{t_0, t_3\}$,

$$\left| |v(t, \cdot)|_{H^{3k-\frac{3}{2}}(\mathbb{R}^n \setminus S_t)}^m - |v(0, \cdot)|_{H^{3k-\frac{3}{2}}(\mathbb{R}^n \setminus S_0)}^m \right| \leq \int_0^t Q dt'$$

for some polynomial Q with positive coefficients. Therefore,

$$E_{ex} \leq \frac{1}{2}E + C_1(1 + |v(0, \cdot)|_{H^{3k-\frac{3}{2}}(\mathbb{R}^n \setminus S_0)}^m) + \int_0^t Q dt' \leq \frac{1}{2}E + C_1 + \int_0^t Q dt',$$

where C_1 is determined only by $|v(0, \cdot)|_{H^{\frac{3}{2}k-\frac{3}{2}}(\mathbb{R}^n \setminus S_t)}$ and the set Λ_0 . Thus

$$E(S_t, v(t, \cdot)) \leq 2E(S_0, v(0, \cdot)) + C_1 + \int_0^t Q dt'.$$

By inserting the above inequality into (4.29) and using proposition 4.1, we obtain (4.4). By choosing μ large enough compared to the initial data, Theorem 4.1 follows. \square

Notation

A^* : the adjoint operator of an operator.

D and ∂ : differentiation with respect to spatial variables.

∇f : the gradient vector of a scalar function f .

∇_X : the directional derivative in the direction X .

\perp and \top : the normal and the tangential components of the relevant quantities.

$\mathbf{D}_t = \partial_t + v^i \partial_{x^i}$ the material derivative along the particle path.

$S_t = \partial\Omega_t$ the boundary of a smooth domain evolving in time.

$N(t, x)$: the outward unit normal vector of S_t at $x \in S_t$.

Π : the second fundamental form of S_t , $\Pi(t, x)(w) = \nabla_w N \in T_x S_t$.

$\Pi(X, Y) = \Pi(X) \cdot Y$.

κ : the mean curvature of S_t , i.e. $\kappa = \text{tr}\Pi$.

$f_{\mathcal{H}} = \mathcal{H}(f)$: the harmonic extension of f on Ω_t .

$\mathcal{N}(f) = \nabla_N \mathcal{H}(f) : S \rightarrow \mathbb{R}$: the Dirichlet-Neumann operator.

$\bar{X} = X \circ u^{-1}$ the Lagrangian coordinates description of X .

\mathcal{D} : the covariant differentiation on $S_t \subset \mathbb{R}^n$.

$\mathcal{D}_w = \nabla_w^\top$, for any $x \in S_t$ $w \in T_x S_t$.

$\mathcal{R}(X, Y)$, $X, Y \in T_x S_t$: the curvature tensor of S_t .

$\Delta_{\mathcal{M}} \triangleq \text{tr}\mathcal{D}^2$: the Beltrami-Laplace operator on a Riemannian manifold \mathcal{M} .

Δ^{-1} : the inverse Laplacian with zero Dirichlet data.

$\Gamma = \{\Phi_{\pm} : \Omega_t^{\pm} \rightarrow \mathbb{R}^n ; \text{ volume preserving homeomorphism, such that } \Phi_+(\partial\Omega^+) = \Phi_-(\partial\Omega^-)\}$

$\bar{\mathcal{D}}$: the covariant derivative on Γ ,

\mathcal{D} : represent $\bar{\mathcal{D}}$ in Eulerian coordinates.

$\bar{\mathcal{R}}$: the curvature operator on Γ .

\mathcal{R} : represent $\bar{\mathcal{R}}$ in Eulerian coordinates.

Π : the second fundamental form of $\Gamma \subset L^2$

$\Pi_u(w_1, w_2) = \nabla_{w_1}^\perp w_2$, for any $u \in \Gamma$, $w_1, w_2 \in T_u \Gamma$

$p_{v,w} = -\Delta^{-1} \text{tr}(Dv Dw)$.

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